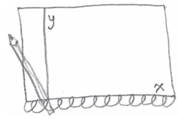
Desktop R^3 Coordinate System 5 min

Later in this chapter we will visualize and graph linear things in R^3 such as lines, vectors, and planes. For now, we start simple, by graphing points in R^3.



1A. Extend good ideas from R^2 to R^3.

1. The back cover of this book represents the nonnegative quadrant of the xy-plane. Use a dry erase marker to draw the x- and y-axes on the clear cover.

2. Insert your pencil in the coil binding or hole to represent the z-axis. You've just accessed the 3rd dimension—your Calculus 3 life begins now.

3. With your marker, plot the point (4, 2) in the xy-plane.

1B. Simplify from R^3 to R^2.

4. Locate the R^3 point $(x, y, z)=(4, 2, 3)$ by moving 3 units directly up from the point you just marked in the xy-plane.

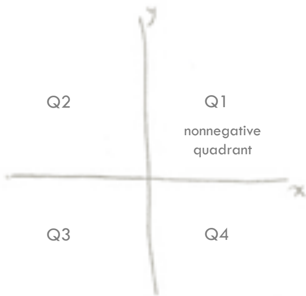
5. Compare this to the R^3 point $(x, y, z)=(4, 2, -3)$. To reach this point, you'd have to punch through the book and move 3 units *down*.

6. Locate these points too.

$(-4, 2, 3)$	$(0, 0, 4)$	$(3, 0, 4)$
$(4, -1, 0)$	$(-1, 0, 2)$	$(0, 3, 3)$

7. Plot the coordinates for one of the following:

your birthday (e.g., March 15, 1993 corresponds to (3, 15, 93)
your graduation day (e.g., Dec. 4, 2009 is (12, 4, 9)
an anniversary



*The 2 axes (x and y) divide the R^2 coordinate system into 4 regions called **quadrants**, which are labeled counterclockwise. Quadrant 1 is also referred to as the nonnegative quadrant.*

How many regions do the 3 axes (x, y, and z) divide the R^3 coordinate system into?

- ❑ 6
- ❑ 8
- ❑ 9
- ❑ 12

While the R^3 **octants** are not labeled, we do often refer to the nonnegative octant, the octant where $x \geq 0$, $y \geq 0$, and $z \geq 0$. Identify the **nonnegative octant** in your Desktop Coordinate System.

Distance d between two R^3 points (x_1,y_1,z_1) and (x_2, y_2, z_2) is given by

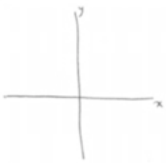
$$d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}\,.$$

The distance between two points in R^3 results from two applications of the Pythagorean theorem. Can you draw a picture that proves this?

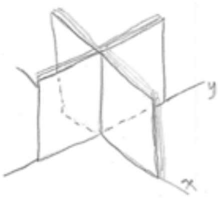
R^3 Coordinate Planes 5 min

There are three coordinate planes in R^3. The set of all points for which z=0 denoted (x, y, 0) creates the familiar coordinate plane, the **xy-plane**, that prior to Calculus 3, was the only coordinate plane you knew. In R^3, there are two others: the **xz-plane** and the **yz-plane**.

1. Draw the x and y axes on a piece of paper.



2. Clump pages of this book as shown.



3. Place the book in the form of #2 (i.e., the intersection of perpendicular walls) over the origin of the xy-plane.

4. Label appropriately: one vertical wall is the xz-plane, the other is the yz-plane. Which is which?

Any R^3 point (x, y, 0) lives in the xy-plane.

- ❑ true
- ❑ false

Any R^3 point with an x-value of 0 lives in the xz-plane.

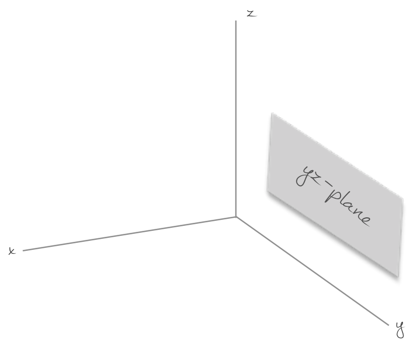
- ❑ true
- ❑ false

	Horizontal	Vertical	
The yz-plane is a	☐	☐	plane.
The xy-plane is a	☐	☐	plane.
The xz-plane is a	☐	☐	plane.

5 min

Corner Coordinate System

A corner in a room (e.g., your room, a classroom, etc.) provides an easy way to visualize the nonnegative octant.

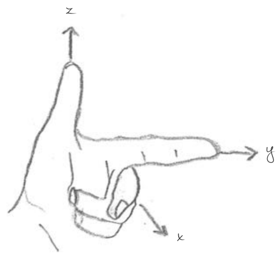


Use post-it notes to label the axes and coordinate planes (i.e., walls and floor) in your Corner Coordinate System.

The xy-plane and the xz-plane intersect at the:

- ❑ x-axis.
- ❑ y-axis.
- ❑ z-axis.

Finger Coordinate System



Your weak (non-writing) hand is a very portable coordinate system.

1. Point the thumb (z-axis) of your weak hand to the sky.
2. Spread your index finger (y-axis) and middle finger (x-axis) so they are about perpendicular to each other and to your thumb.
3. Optional: add tick marks on the axes.
4. Plot the point (4, 1, 3).

Planes with 2 free variables 3 min

Let's begin with the easiest R^3 functions to visualize and graph. R^3 planes such as z=2 or x=1 or y=3 restrict only one variable at a time, leaving the other two variables "**free.**"

Visualize the R^3 function z=f(x, y)=2 with your Desktop system.

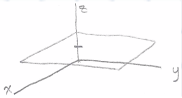
1. Go to the point z=2 on the z-axis; (0, 0, 2).

2. Because x and y are not mentioned, they are "free," free to take on any value from $-\infty$ to ∞. Start with x. Since x is free, the point on the z-axis (0, 0, 2) becomes a **line** moving parallel to the x-axis.

3. Now y is also free which makes this line extend in the y-direction, creating a **plane**.

The z=2 plane is a horizontal plane and is parallel to which coordinate plane?
- ❑ xz-plane
- ❑ xy-plane
- ❑ yz-plane

ANSWER



***z=2 plane**. Planes extend infinitely so just draw a section of them. This plane extends in the x and y directions, so it helps to copy the angles created by the x- and y-axes.*

Plane with 1 free variable

7 min

The next easiest R^3 functions to visualize and graph are R^3 planes with 1 free variable, such as $z=x$, $y=x$, $y=3x-4$, and $z=-2y$.

Visualize the R^3 function $x + y = 4$ here with this page.

STEP 1

The R^2 line $x+y=4$ is graphed below. This page is the xy-plane.

1B. Simplify from R^3 to R^2.

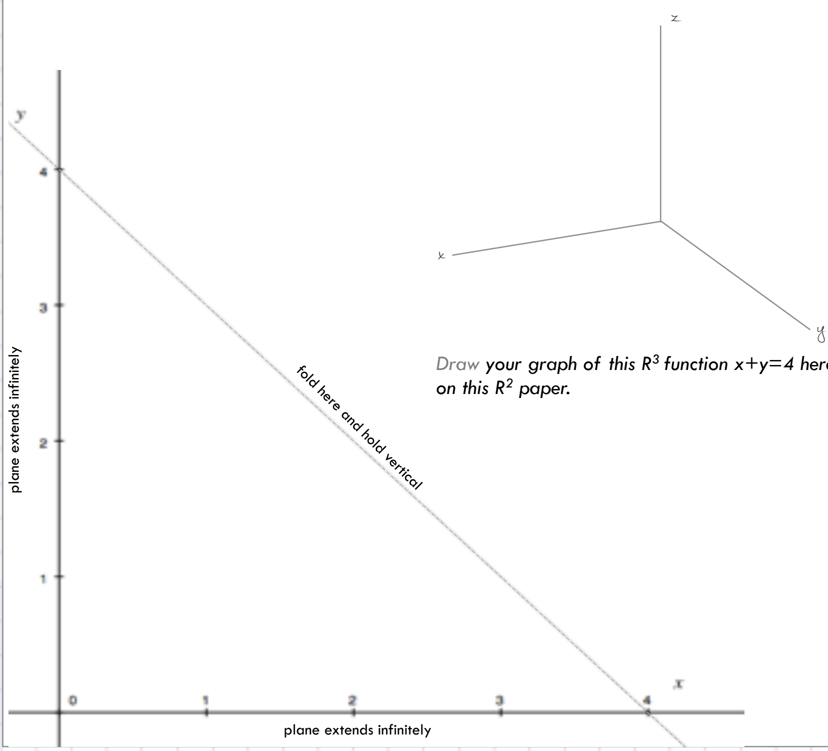
STEP 2

Since z is not mentioned, it is *free* to take on any value from $-\infty$ to ∞. So this 2-D line extends up and down, creating a 3-D vertical plane.

STEP 1

Fold the corner of this page up creating a section of the vertical R^3 plane $x+y=4$.

1A. Extend good ideas from R^2 to R^3.



Draw your graph of this R^3 function $x+y=4$ here on this R^2 paper.

z

upright pencil here creates y-axis

0 1 2 3

x

Use this page to visualize the R^3 function

$z = -6 + 3x.$

1. Fold the page up along the digaonal line revealing the R^3 plane $z = -6+3x$.

2. Recover the usual orientation of R^3 by holding the book upright so the positive z-axis is pointing up.

This page is the ***xz-plane***.

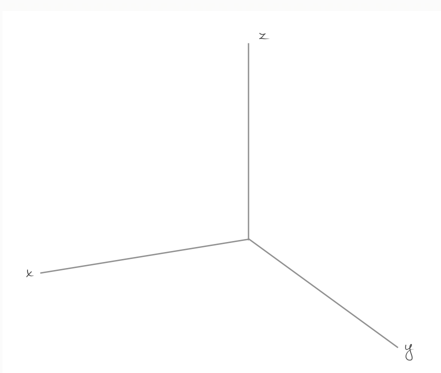
-3

This line is the line $z = -6+3x$ in the xz-plane.

-4

-5

-6



-7

Draw your graph of the R^3 function $z = -6+3x$ on this R^2 paper here.

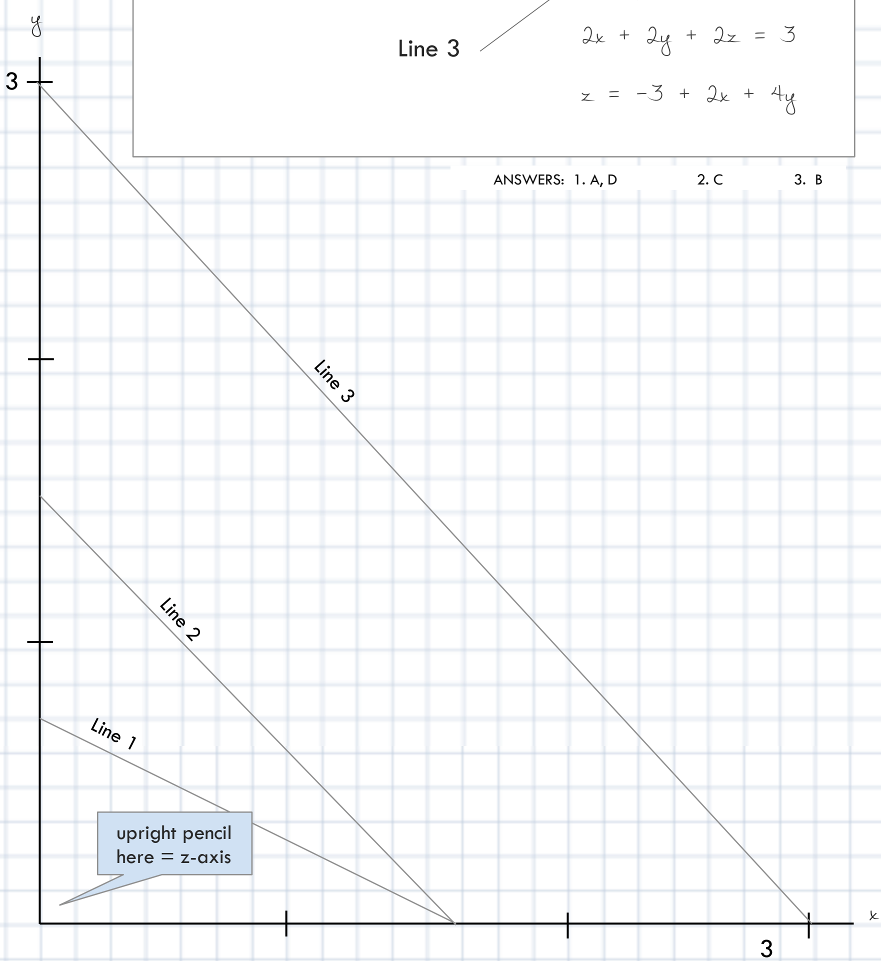
Cutting Planes

4 min

Planes extend infinitely. Since we can't draw infinitely, then, of course, we draw just a section of the plane. All but one of the following four planes **cuts through** the nonnegative octant of R^3. Fold the corner of the page along each line and match the line to its corresponding expression as a plane.

Line	Plane
Line 1	$z = 3 - 2x - 4y$
Line 2	$x + y + z = 3$
Line 3	$2x + 2y + 2z = 3$
	$z = -3 + 2x + 4y$

ANSWERS: 1. A, D 2. C 3. B



Graphing nonlinear R^3 functions

that have 1 free variable

3 min

So far we've graphed the simplest type of R^3 functions, flat **linear** objects called planes. **Nonlinear** objects are even more interesting. Begin by graphing the easiest type of nonlinear R^3 functions, those with 1 free variable. Specifically, graph the R^3 function $y = x^2$ in your Desktop System.

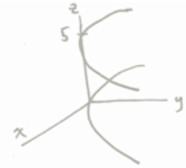
1. Graph $y=x^2$ in the xy-plane.

2. z is free so extend the R^2 graph in the xy-plane vertically up and down.

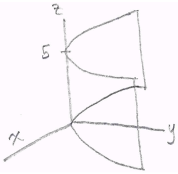
3. Bend a piece of paper to visualize the surface created by the R^3 function $y=x^2$.

STEPS TO GRAPHING THE 3-D SURFACE ON 2-D PAPER

1. *Draw one copy of the 2-D parabola when z=0.*



2. *Draw another copy of the 2-D parabola when, say, z=5.*



3. *Connect the matching ends of the two parabolas.*

Visualize the R^3 function $z = e^{x-2}$

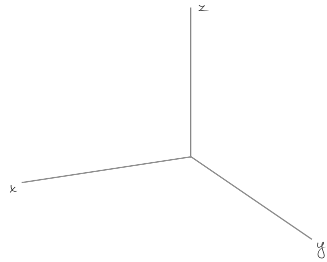
7 min

METHOD 1: Desktop Coordinate System

1. Graph $z=e^{x-2}$ in the xz-plane. This time the back cover is the xz-plane.

2. y is free so extend the R^2 graph along the y-axis. Use paper to represent this R^3 surface.

3. Rotate the entire coordinate system so that it has the usual orientation (with the positive z-axis pointing up).



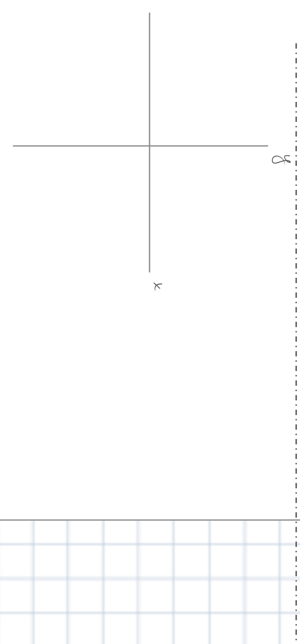
Graph, on this R^2 paper, the R^3 function $z = e^{x-2}$ that you just visualized in 3-D.

METHOD 2: Manipulate this page

1. Locate the origin on xy-plane shown to the right. Imagine a z-axis pointing up.

2. Tear the page along the dashed line.

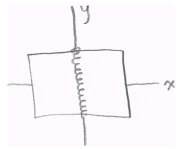
3. Bend the lower half of the page up creating the exponential ramp formed by $z=e^{x-2}$.



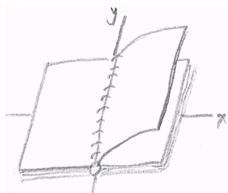
Paper Surfaces

15 min

Lay this book open. This is the xy-plane.



Use your hands to manipulate this page and the previous page to create each surface below. For example, the pages of this book visualize the R^3 surface $z=\sqrt{x}$.



book pages as $z = \sqrt{x}$

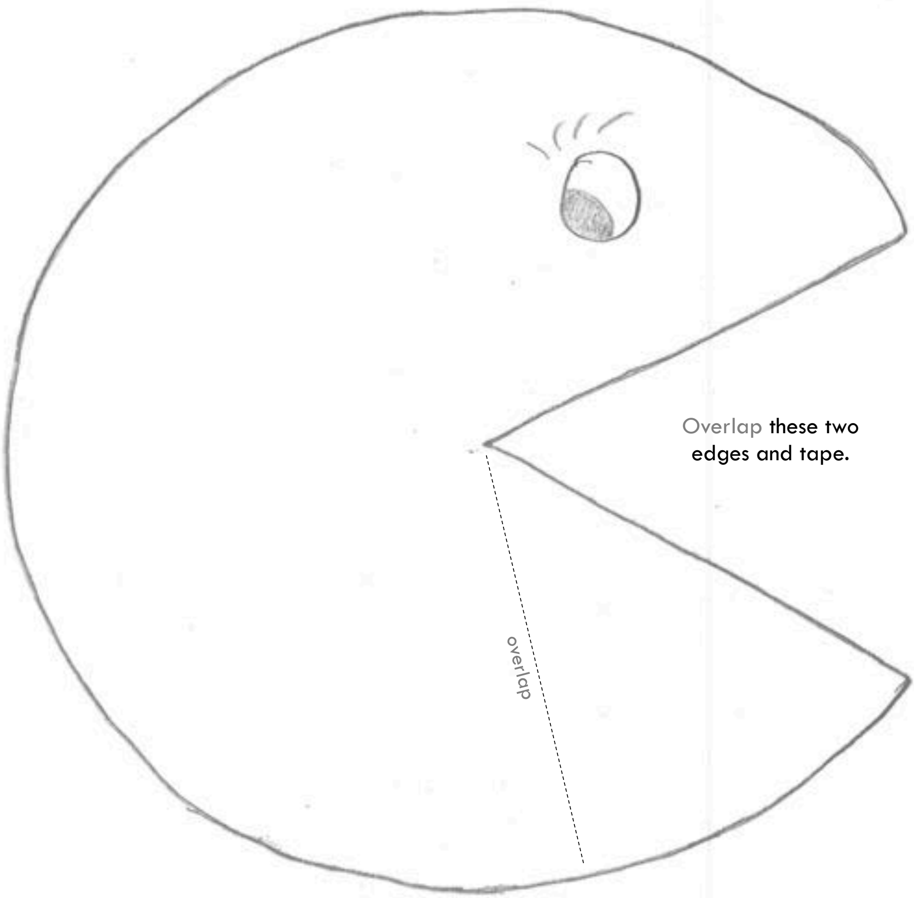
1. $z = x^2$
2. $z = x^3$
3. $z = |x|$
4. $z = e^x$
5. $z = \ln(x+1)$
6. $z = \sin x$

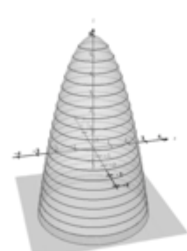
How does each surface above change if x is replaced by y?

- ❑ rotated 90°
- ❑ stays same

ANSWERS: next page

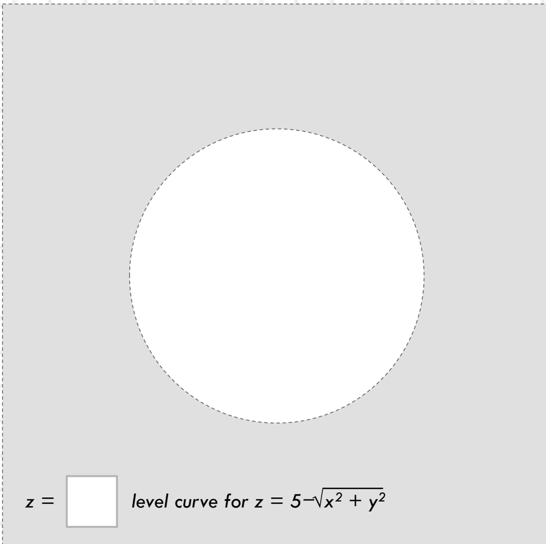
Cut out **Pac-man.** Follow the instructions to make a cone. You'll need this Pac-man cone for the next few pages.

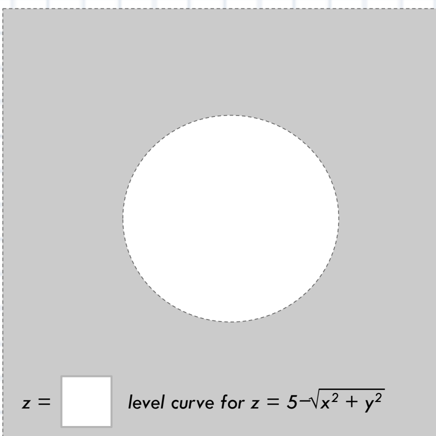


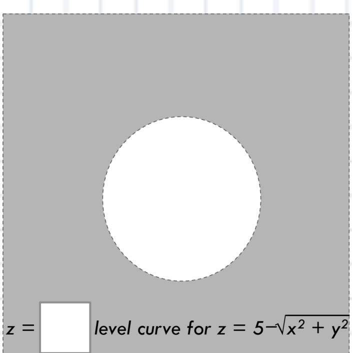


Art: Calculus Desk Ornament

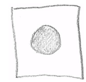
8 min







1. Cut out the 3 squares and the hole out of the middle of each.



2. Slide each center-less square over the inverted Pac-man cone until it fits snug.

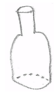
3. Estimate the value k of the z = k horizontal plane that corresponds to each level curve. Notice that each circular level curve is the intersection of the z=k plane and the R^3 function $z = 5 - \sqrt{x^2 + y^2}$. *The z = k planes slice through the R^3 function like an egg slicer through a hard-boiled egg.*

4. Make a Calculus Desk Ornament by adding color to your tree, either the cone or the z=k planes. Add more planes of other shapes besides squares such as hexagons or rectangles.

Bottle Traces

10 min

1. Find an empty *clear* plastic bottle. (Chug a drink from a vending machine.) Sit the bottle upright on your desk. If you can't find a bottle, the *thought experiment* is useful.



2. Use a marker to draw about 8 uniformly spaced z=k traces on the bottle.

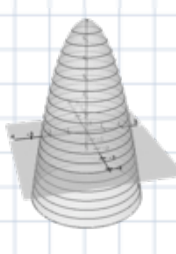
3. Get a birds-eye view over the bottle. Imagine yourself sitting in a crow's nest way up the z-axis, shining a light and projecting each level curve onto the xy-plane. This should help you draw the contour map associated with the bottle.

4. Repeat with other clear objects (e.g., plastic wine glasses from a dollar store, empty glass jars from the recycle bin).

5. BONUS: Repeat with other everyday 3-D objects (phone, wallet, pen, shoe, hat, …). Draw the level curves and contour map each time you hold the object *at a different angle or orientation.*

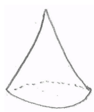
6. DOUBLE BONUS: Now find an edible 3-D object (an unpeeled banana, an unwrapped cupcake, a hard-boiled egg, …). Draw the level curves on the object and draw the contour map. Now eat part of the object and repeat.

Level curves are also called **traces** because they trace the intersection of the $z = f(x, y)$ function and the $z = k$ level curve and project it onto the xy-plane of the contour map.

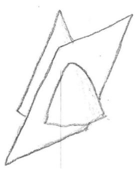


How Greeks got curves from cones

8 min



Set your Pac-man cone tip up. (Make another cone if, alas, Pac-man is gone.)



*The ancient Greeks created their favorite 2-D curves by intersecting a 3-D cone with certain planes.**

Match the 2D-curve in the left column with the plane in the right column that created it by intersecting the Pac-man cone.

2-D Curves	**Plane**
circle	$z = \frac{1}{3} x + k$
ellipse	$z = k$
parabola	$x = k$

* *Mathematician's Note*: The Greeks had to think about 2-D curves in this way since algebra was not yet invented. They could not think about a parabola as $y=x^2$ and graph it. This luxury came thousands of years later with the invention of *analytic geometry*. For more on the marriage of algebra and geometry, see the graphic novella, "Descartes and the Fly," at the end of this chapter.

Topo Maps are Contour Maps

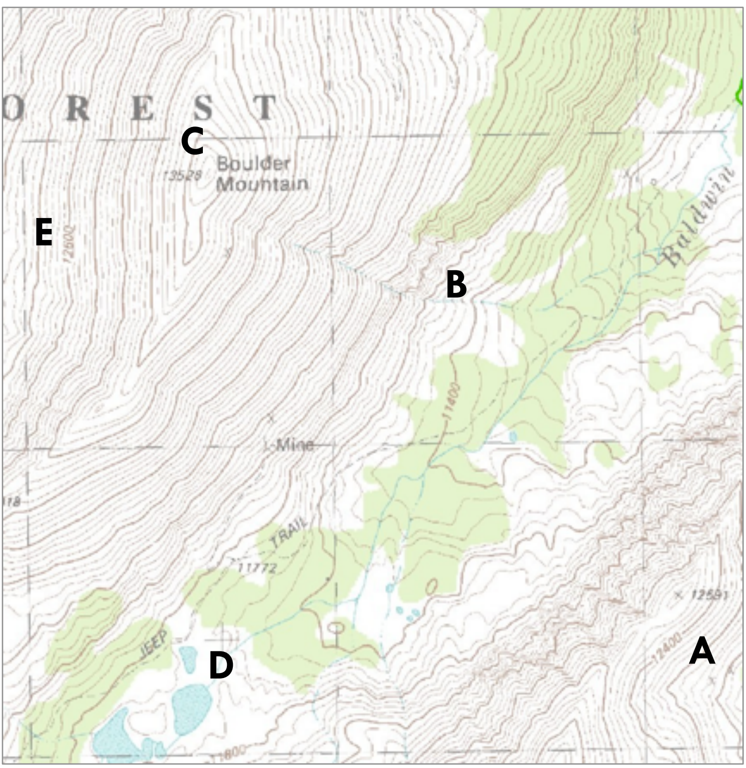
7 min

1C. Think geometrically.

1. Describe the hike from point A to B to C. Is it uphill or downhill? Steep or shallow?

2. Jack is at point E. Can he see Jill at point A? B? C? D?

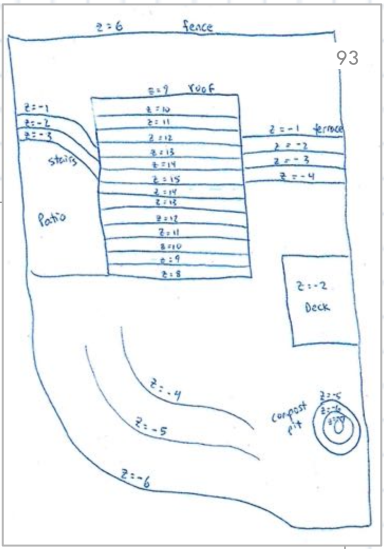
3. Find the easiest route to hike from C to D. Is there a difference between the easiest route and the most direct route?



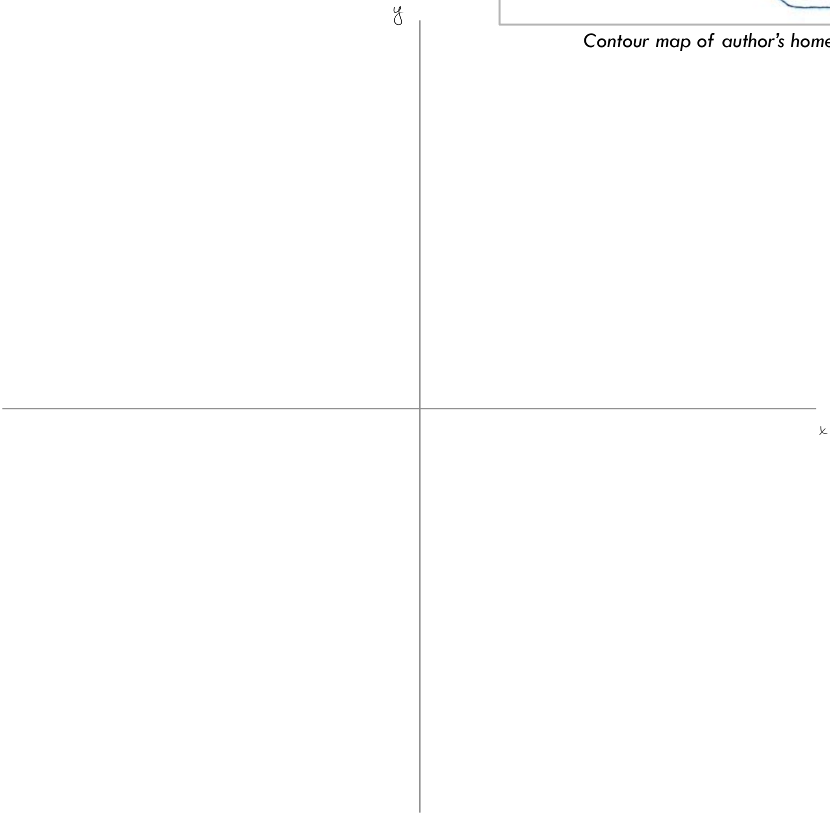
Topo a property 12 min

The image on the right is a contour map of my home. Draw a contour map for a property that's familiar to you (e.g., your home, a favorite park, etc.).

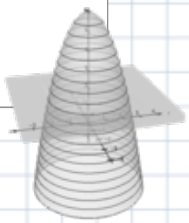
1C. Think geometrically.



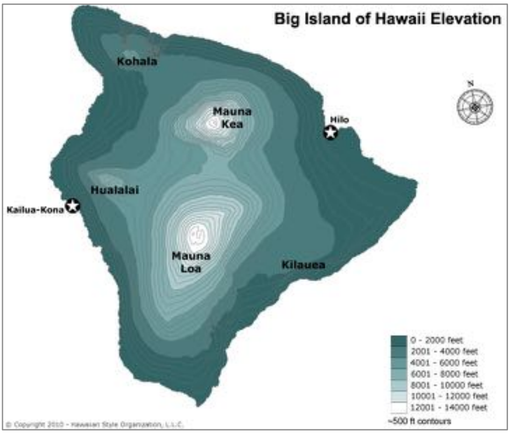
Contour map of author's home



Contour map of ______________________



Art: Shaded Topo Map

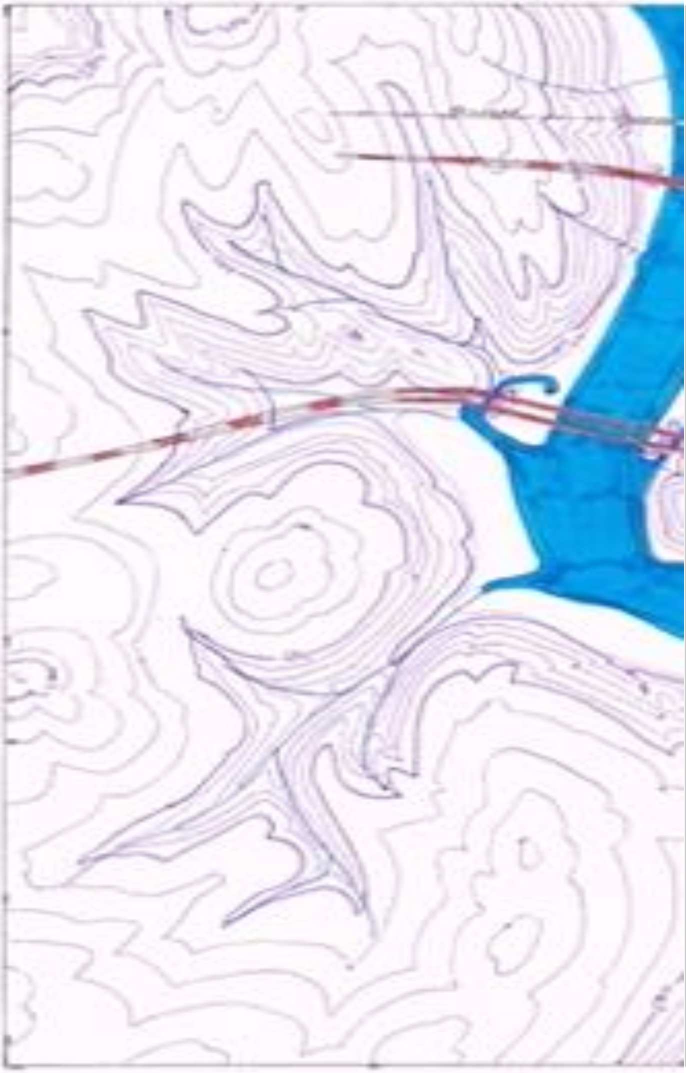


Some topographical maps ***shade bands in ranges of elevations*** *according to a legend as shown in this example map of Hawaii.*

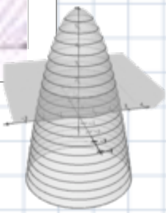
mollymaps.com

This shading technique can also create beautiful artwork such as that of artist-geographer Molly Holmberg Brown.

Try this shading technique to add your own artistic flair to Michael Smith's wonderful hand-drawn map.



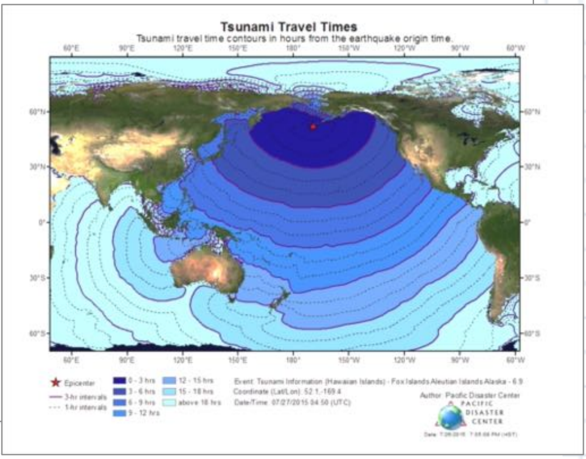
https://easky30.files.wordpress.com/2013/02/topo-map.jpg



Contour Map Collection

15 min

Use this page to collect other cool contour maps such as nautical charts that show sea depths, population growth maps, . . .



dynamic.pdc.org

Further Research: travel time maps, isobath lines, and isohyet lines.

Topo your fist

5 min

Ball your weak hand into a fist, palm down. Your fist is an R^3 surface and each knuckle becomes a small peak.

Use your dominant hand to draw contour lines on your weak hand. Remember $z = k$ level curves are horizontal planes that slice through your fist surface.

Now flatten your hand on the desk to reveal the contour map.

1C. Think geometrically.

Discuss with a friend the accuracy of the contour map. Are there any distortions introduced by flattening your hand?

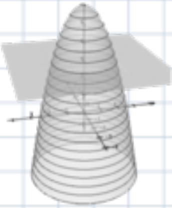
ANSWER

1B. Go from R^3 to R^2.

3-D Fist Surface

2-D Fist Contour Map

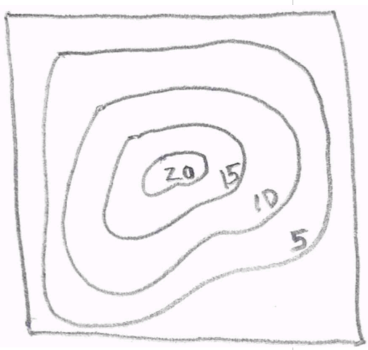
http://www.scouterlife.com/2012/05/topographic-map-activity.html?m=1

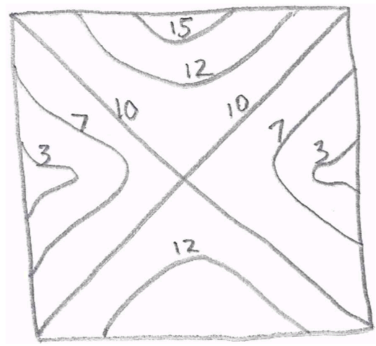


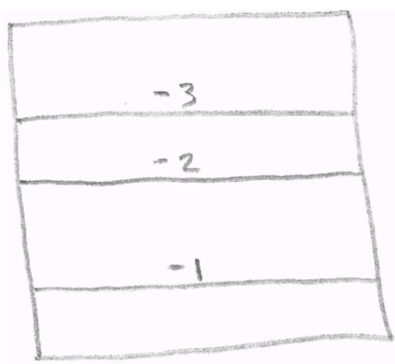
TopoSync

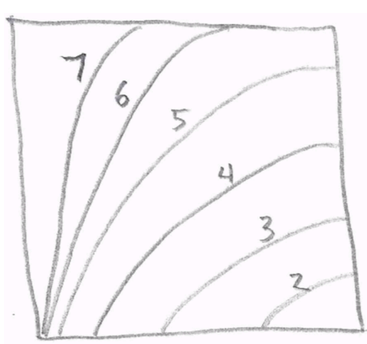
10+ min

Add contour lines with z=k values to the missing t-shaped cross-section to create logical smooth transitions and/or interesting surfaces. *Bonus:* Cut out the four little topo maps below. Rotate and rearrange them and play again and again.





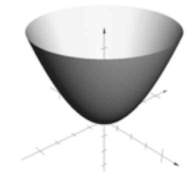




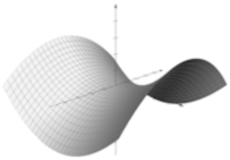
Contour Match I

7 min

Match each surface on this page with a contour map on the next page.

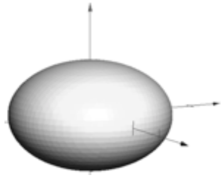


1. *circular paraboloid*
   $z = x^2 + y^2$

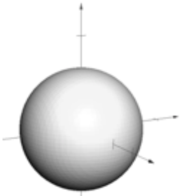


2. *hyperbolic paraboloid*
   $z = x^2 - y^2$

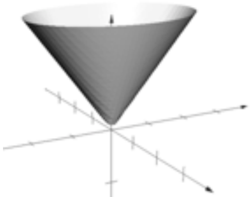
   Further Research: *double hyperbolic paraboloid house in Lawrence, Kansas*



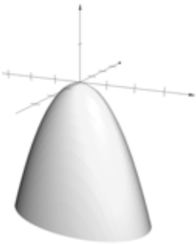
3. *ellipsoid*
   $ax^2 + by^2 + cz^2 = d$



4. *sphere*
   $x^2 + y^2 + z^2 = r^2$



5. *elliptic cone*
   $z = \sqrt{ax^2 + by^2}$

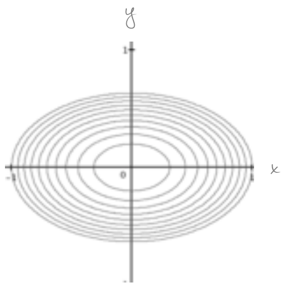


6. *elliptic paraboloid (opening down)*
   $z = -(ax^2 + by^2)$

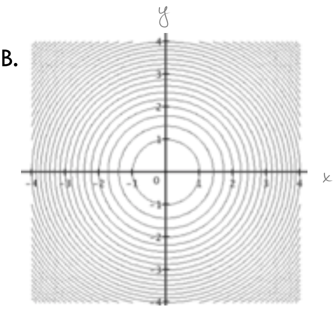
ANSWERS: 1. B 2. C 3. A 4. D 5. F 6. E

Match each contour map on this page with a surface on the previous page.

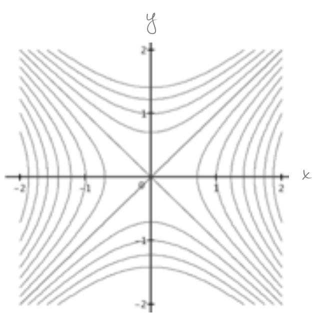
A.



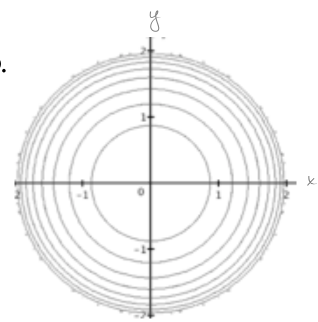
B.



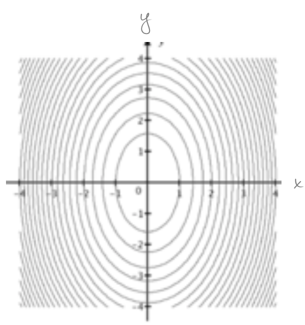
C.



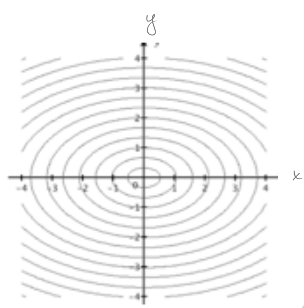
D.

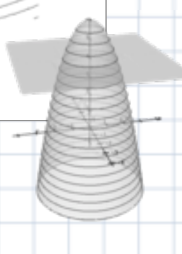


E.



F.



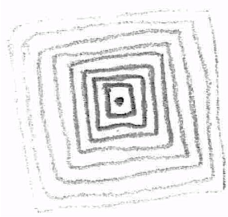


Contour Match II

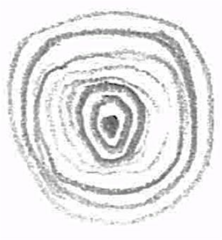
7 min

Match each contour map on this page with a surface on the next page.

1.

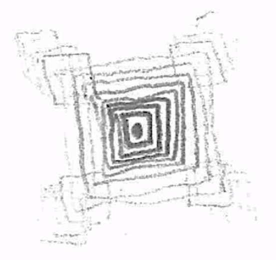


2.



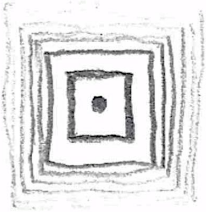
3.

4.



5.

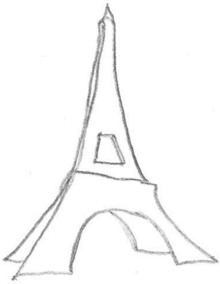
6.



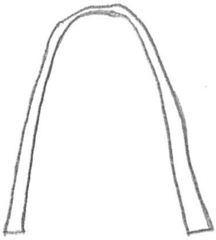
ANSWERS: 1. F 2. E 3. C 4. B 5. D 6. A

Match each surface on this page with a contour map on the previous page.

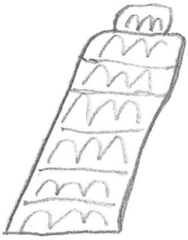
A. *Washington Monument*



B. *Eiffel Tower*

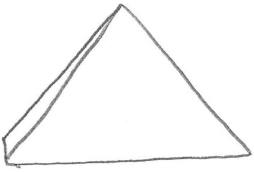


C. *St. Louis Arch*

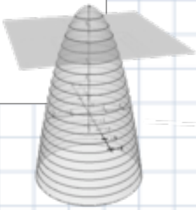


D. *Leaning Tower of Pisa*

E. *Seattle Space Needle*



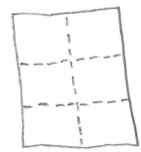
F. *Great Pyramid*



Transparency Game

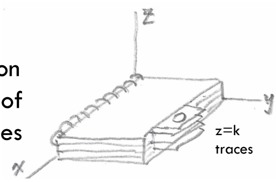
15 min

Constructing the slices that create a 3D surface helps to understand contour maps and level curves.



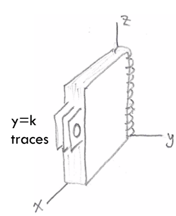
1. Cut the transparency page of the extra clear back cover into 6 rectangles. Each will represent a z=k level curve and will be inserted in evenly spaced intervals between the pages of the book.

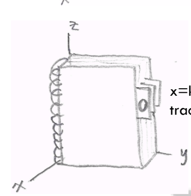
2. Pick a surface from the Surface Gallery at the end of the book. Start with a simple surface such as a sphere or cone. With a transparency marker, draw a level curve on each transparency and insert in the edge of the book. Viewed from above, these 6 slices of the surface help you see the 3D object.

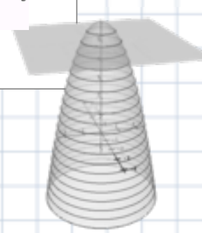


3. Clean the transparencies. Pick another surface and play again (with a partner).

4. Variations: When the journal lays flat on a desk, each transparency represents a z=k trace. When the journal stands upright, each transparency represents either an x=k trace or a y=k trace depending on the orientation of the upright journal. Repeat the game selecting a surface (e.g., a cone, a sphere) and draw its x=k traces. Then its y=k traces.



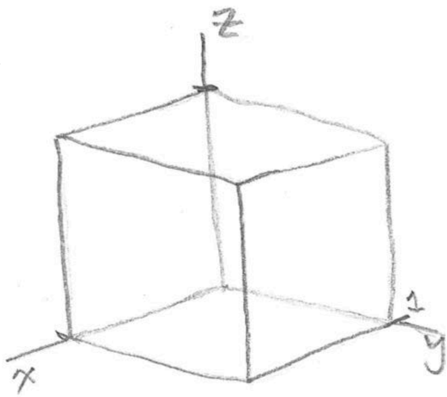




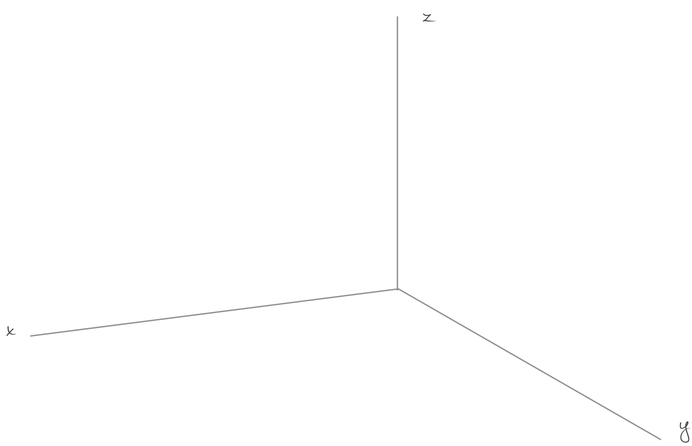
Cut the Foam

10 min

Use a block of foam (or cheese) to represent the unit cube you visualized with the six half-space inequalities: $x \geq 0$, $x \leq 1$, $y \geq 0$, $y \leq 1$, $z \geq 0$, $z \leq 1$.



Now cut the foam to represent the region given by the intersection of those same six inequalities with these two additional inequalities: $x + y + z \geq 1$ and $x + y + z \leq 2$.



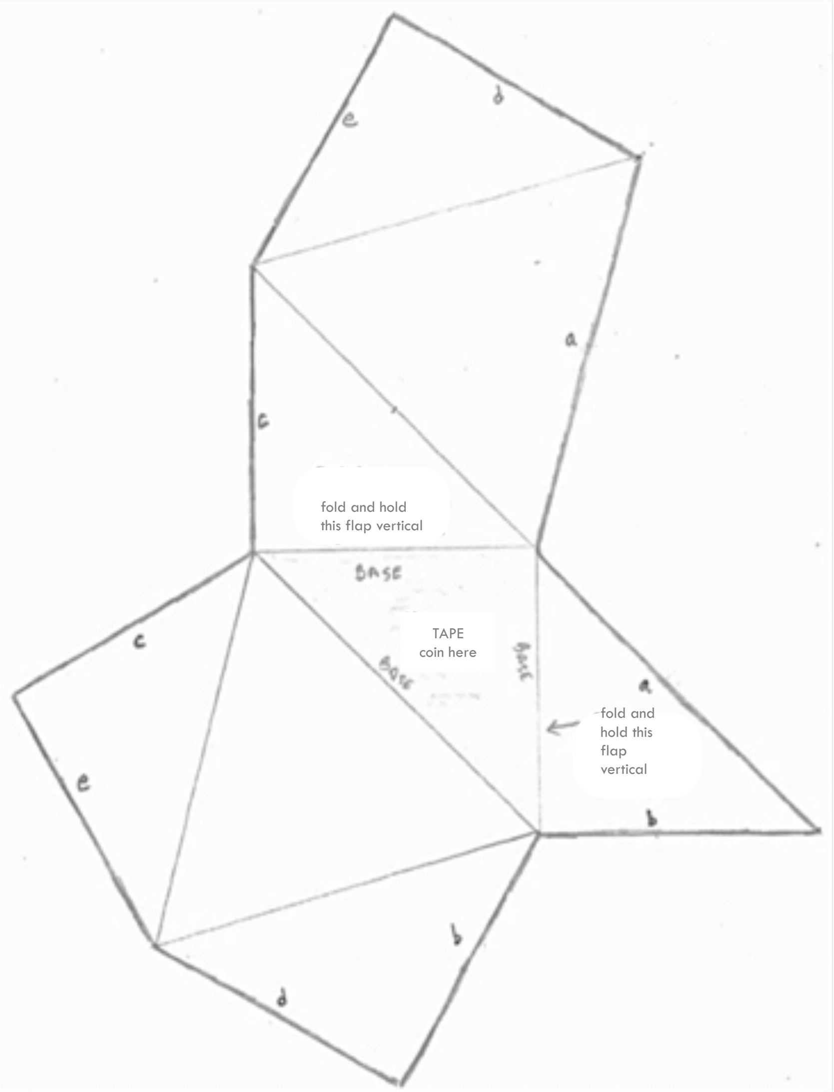
Draw, on this R^2 *paper, the* R^3 *foam region that you just created by intersecting the eight half-space inequalities:* $x \geq 0$, $x \leq 1$, $y \geq 0$, $y \leq 1$, $z \geq 0$, $z \leq 1$, $x+y+z \geq 1$, $x+y+z \leq 2$.

Do you see all the same **faces** on the drawing that you see on the foam? On the foam, identify the inequality that creates each face.

Answer: foam region

3 min

Compare your foam region to the correct region, by constructing it origami style. Cut along the darkened outer edge of the shape. Fold as directed and tape edges with matching labels.



*A **calculus paperweight** created by the intersection of eight half-spaces, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, $1 \le x+y+z \le 2$.*