Plane curve or space curve? 4 min

Next to each item, place a **p** (for plane curve), **s** (for space curve), or **n** (for neither).

1. path of raindrop on inverted V-roof
2. path of ant on concrete floor
3. hiking trail
4. path of ant on ground
5. monorail course

6. path of drone
7. commute from home to work
8. path of leaf in tornado
9. path of leaf in river
10. path of leaf in mountain creek

11. path of fly
12. path of skijumper
13. path of skydiver
14. path of ball on pool table
15. blanket over bed

16. charging cable laid over shoes
17. mountain bike trail
18. path of golf putt
19. cable on suspension bridge
20. laundry line

21. telephone wire
22. roller coaster course
23. path of walking dog
24. Olympic luge course
25. climbing rope with climber and belayer

ANSWERS: 1-5. ppsss 6-10. sssss 11-15. ssspn 16-20. ssspp 21-25. pssss

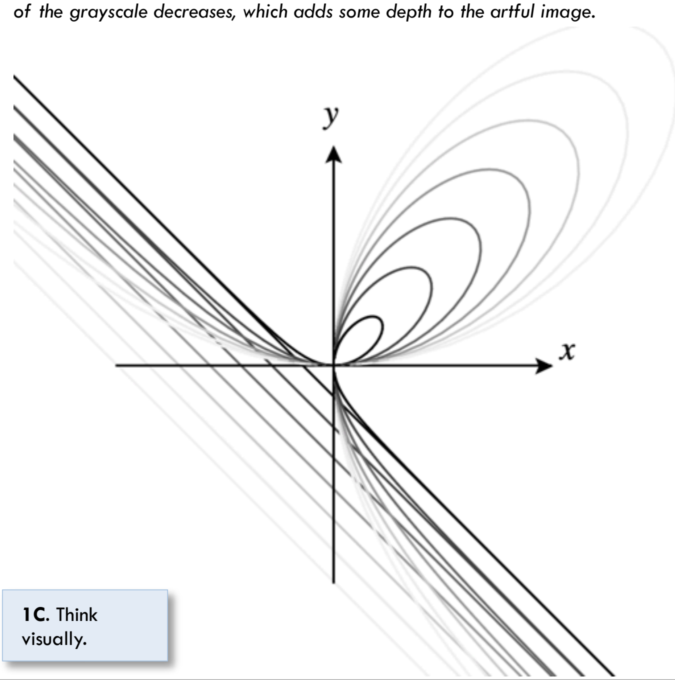
Art: not-so-plain curves

15 min

Research a few of the famous plane curves listed below. Most of these plane curves have one or two parameters (e.g., **a** or **b**) that the user can set to alter the shape of the curve. Play with a few values for the parameters, then graph with a tech tool, such as CalcPlot3D, on the same screen, several variations of the same plane curve with various values for the parameter(s). The goal is to make something visually interesting with these mathematical objects. *Bonus*: Print your work, clip it to this page, and shade your plane curves artistically.

- Witch of Agnesi
- Rose curve of Grandi
- Folium of Descartes
- Epicycloid
- Prolate cycloid
- Conchoid of Nicomedes
- Spiral of Archimedes
- Spiral of Lituus
- Spiral of Euler
- Lemniscate of Gerono
- Lissajous curve
- Harmonograph

*Example: **Folium of Descartes,** $r(t)=\langle 2at/(1+t^3),\ 2at^2/(1+t^3)\rangle$ for $-12\le t\le 12$, with several different values of **a**, namely **a**=1, 2, 3, 4, 5, 6, 7. As **a** increases, the darkness of the grayscale decreases, which adds some depth to the artful image.*



1C. Think visually.

Plane Curve Match

6 min

Match each plane curve to an image.

1. $r(t)=\langle t, \pi - t\rangle$ for $0 \le t \le 2$

2. $r(t)=\langle \tfrac{1}{2}t, \pi - \tfrac{1}{2}t\rangle$ for $0 \le t \le 4$

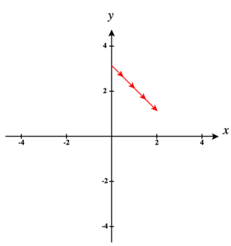
3. $r(t)=\langle 3\cos t, 4\sin t\rangle$ for $0 \le t \le 2\pi$

4. $r(t)=\langle 4\cos t, 3\sin t\rangle$ for $0 \le t \le 2\pi$

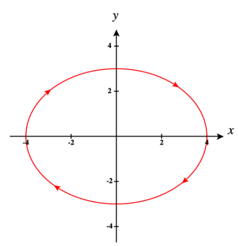
5. $r(t)=\langle \cos t + t\sin t, \sin t - t\cos t\rangle$ for $0 \le t \le 6\pi$

6. $r(t)=\langle 3\cos t, -4\sin t\rangle$ for $0 \le t \le 2\pi$

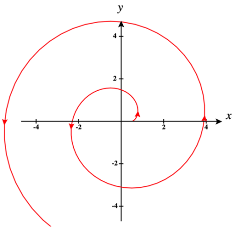
A.



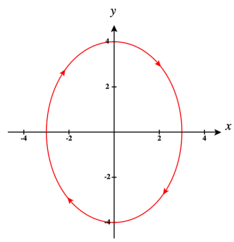
D.



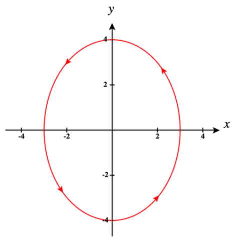
B.



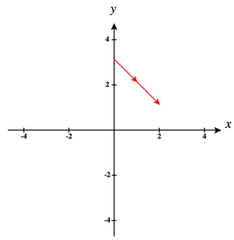
E.



C.



F.



ANSWERS: 1. F 2. A 3. C 4. D 5. B 6. E

Space Curve Match I

6 min

Match each space curve to an image, then to a verbal description.

1. $r(t)=\langle t, t, t^2\rangle$ for $0 \leq t \leq 2$

2. $r(t)=\langle t^2, t, t\rangle$ for $0 \leq t \leq 2$

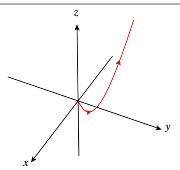
3. $r(t)=\langle t, t^2, 0\rangle$ for $0 \leq t \leq 2$

4. $r(t)=\langle t, t^2, 2\rangle$ for $0 \leq t \leq 2$

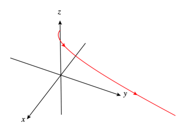
5. $r(t)=\langle 1, 1, t^2\rangle$ for $0 \leq t \leq 2$

6. $r(t)=\langle t, t, 2t\rangle$ for $0 \leq t \leq 2$

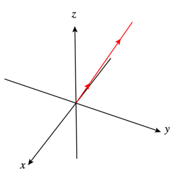
A.



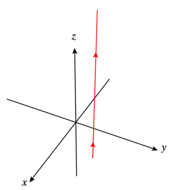
D.



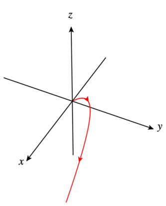
B.



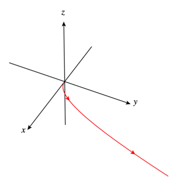
E.



C.



F.



I. parabola in $y=z$ plane

II. parabola in xy-plane

III. parabola in $y=x$ plane

IV. parabola in $z=2$ plane

V. line through origin

VI. vertical line

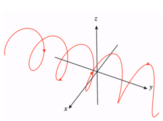
ANSWERS: 1. A, III 2. C, I 3. F, II 4. D, IV 5. E, VI 6. B, V

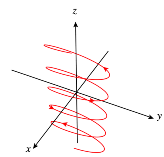
Space Curve Match II

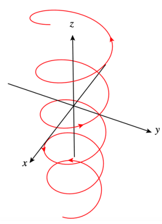
6 min

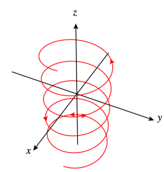
Match each space curve to an image, then to a verbal description.

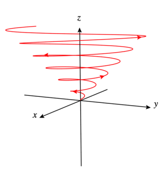
1. $r(t)=\langle 4t\cos t,\ 4t\sin t,\ 4t\rangle$ for $0 \le t \le 12\pi$
2. $r(t)=\langle 4\cos t,\ 4\sin t,\ t\rangle$ for $-6\pi \le t \le 6\pi$
3. $r(t)=\langle 4\cos t,\ 4\sin t,\ \tfrac{1}{2}t\rangle$ for $-6\pi \le t \le 6\pi$
4. $r(t)=\langle \cos t,\ 4\sin t,\ \tfrac{1}{2}t\rangle$ for $-6\pi \le t \le 6\pi$
5. $r(t)=\langle 4\cos t,\ 4\sin t,\ 0\rangle$ for $-6\pi \le t \le 6\pi$
6. $r(t)=\langle 4\cos t,\ t,\ 4\sin t\rangle$ for $-6\pi \le t \le 6\pi$

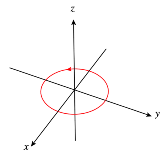
A.


D.


B.


E.


C.


F.


I. repeated circle

II. helix on lying cylinder

III. helix on cone

IV. tighter helix on cylinder

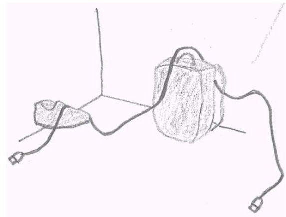
V. helix on elliptic cylinder

VI. helix on cylinder

ANSWERS: 1. C, III 2. B, VI 3. E, IV 4. D, V 5. F, I 6. A, II

Roller Coaster

5 min



1. Design a roller coaster by laying a charging cable over some stuff (e.g., shoes, backpack).

2. Decide on an orientation of travel. Then pick a point t on the roller coaster.

3. Pick another point t+h a bit further downtrack from t.

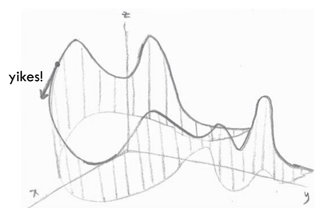
4. Visualize the secant vector between points t and t+h.

5. Cut h in half and repeat 4. Repeat the halving of h and visualizing of the new secant vector a few more times.

6. As h→0, the secant vectors converge to the
   - ☐ tangent vector at t+h
   - ☐ tangent vector at t
   - ☐ secant vector at t

7. So the tangent vector at any point t gives the direction of motion. Reverse the orientation of your coaster. Visualize the tangent vector at the same point t. How does it relate to the tangent vector for the original orientation? Now visualize the tangent line. Is the tangent line at point t the same whether you ride forward or backward on the coaster?

8. Finally pick one last point on your coaster. Imagine you are riding on the coaster and your flip flop flies off at this point. Where do you go to look for it?

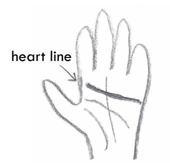


Moral: *For any smooth vector function r(t) the tangent vector at time t gives the direction of travel if the imaginary roller coaster were to derail at precisely point t.*

Heart line

5 min

1. Find the heart line of your weak non-writing hand. This is a:
   - ❑ plane curve.
   - ❑ space curve.



palm of hand

2. Decide on an orientation of travel along this heart line curve (e.g., towards either the pinky or thumb). Mark a point on this curve.

3. Visualize the tangent vector at this point.

4. With a pen draw on your hand to extend this curve creating a much longer, more interesting, highly nonlinear space curve. Think of this as henna, Calculus 3-style.

extending the space curve to back of hand

5. Find a point on this extended heartline curve that is taller than its neighbors. The tangent line there is:
   - ❑ horizontal.
   - ❑ vertical.
   - ❑ neither.

6. Find a point on this extended heartline curve that is smaller than its neighbors. The tangent line there is:
   - ❑ horizontal.
   - ❑ vertical.
   - ❑ neither.

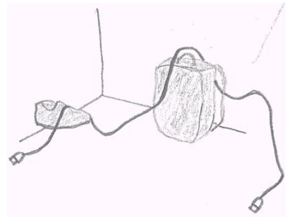
7. What can we say about the derivative r'(t) at such points?
   - ❑ r'(t)=<0, 0, 0>
   - ❑ r'(t)=<x'(t), y'(t), 0>
   - ❑ r'(t)=<0, y'(t), 0>

8. If r'(t)=<x'(t), y'(t), 0>, then the point t must be a local extrema in terms of its height.
   - ❑ true
   - ❑ false

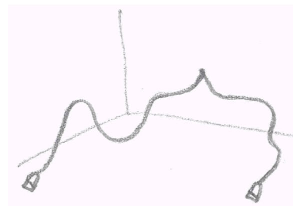
Danger Coaster

4 min

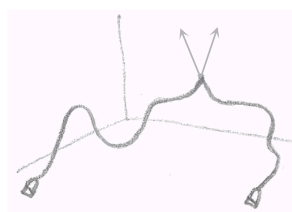
1. Design a roller coaster by laying a charging cable over some stuff (e.g., shoes, backpack). Is the coaster r(t) smooth everywhere?
   - ❑ yes
   - ❑ no



2. Pick a point on the roller coaster. Call it c. Manipulate the cable so that this point is now non-smooth. Which two of the following manipulations of the cable make r(t) non-smooth at c?
   - ❑ twisting the cable into a loop at c
   - ❑ pinching the cable into a point or cusp
   - ❑ making an abrupt angular change in the cable



3. At this cusp point c, $r'(c)=<0, 0, 0>$ and the coaster stalls (there is no change in any direction, not x, not y, not z) for just a tiny moment as it makes the abrupt turn. Tangent vectors help visualize why r(t) is non-differentiable at c. Ride the coaster in one direction and imagine derailing at c. Now ride in the reverse direction and imagine derailing at c. At the non-smooth point c, the two tangent lines of derailment are:
   - ❑ the same.
   - ❑ different.



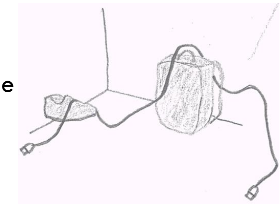
Moral: *At the non-smooth point c, r(t) is said to be non-differentiable at c because the tangent vectors differ depending on the orientation of travel. More technically, the limit of the secant vectors as h approaches 0 from the right differs from the limit of the secant vectors as h approaches 0 from the left.*

4. Check a few other smooth points. At smooth points, do the two directions of derailment lie on the same tangent line?
   - ❑ yes
   - ❑ no

Motorcycle Normals

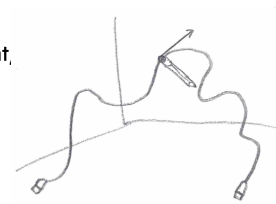
4 min

1. Lay a charging cable over some stuff (e.g., shoes, backpack). This time the cable is a road that you are motorcycling along. Decide on an orientation of travel.



2. Pick a point at or near the curviest part of the road. Visualize the tangent vector there. This gives the direction of motion precisely at that point.

3. Place your pencil to indicate a vector that is orthogonal to this tangent vector. At this point, how many vectors are orthogonal to this tangent vector?
   - ❑ 1
   - ❑ 2
   - ❑ infinitely many



4. Place your hand to visualize the **normal plane** consisting of the infinitely many normal vectors orthogonal to the tangent vector.

5. We are interested in just one of the vectors in this normal plane. It is called *<u>the</u> normal vector because it is the one vector in the normal plane that gives information about the direction that the road is turning precisely at the point of tangency.* Place your pencil to indicate **the** normal vector. It must: (1) be normal to the tangent vector, (2) be in the normal plane, and (3) point toward the road's curve.

6. Now imagine that you are motoring twice as fast along the same road, so there is a new parametrization $r_2(t)$. Between the two road functions $r(t)$ and $r_2(t)$, do the tangent and normal vectors at any point on the curve change? Suppose you reverse the orientation. Do the tangent and normal vectors change from the original $r(t)$ to the new $r_2(t)$?

Moral: *At a particular point, the tangent vector gives the direction of motion along the curve $r(t)$, i.e., the direction the motorcycle would travel were you to lose steering. On the other hand, the normal vector gives the direction of the turn of the road $r(t)$. You must lean the motorcycle this direction while in this turn of the road.*

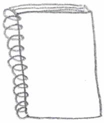
1C. Think visually.

Coil

5 min

1. Set this journal upright so that the coil represents a vertical helix.

2. Which vector function below represents this vertical helix?



- ❑ r(t)=<t, cos(t), sin(t)>
- ❑ r(t)=<t, sin(t), cos(t)>
- ❑ r(t)=<cos(t), sin(t), t>
- ❑ r(t)=<cos(t), t, sin(t)>

3. Which set could represent the domain?

- ❑ $0 \leq t \leq \infty$
- ❑ $0 \leq t \leq 10$
- ❑ $-\infty \leq t \leq \infty$

4. The normal vector at every point t points toward the interior of the coil.

- ❑ true
- ❑ false

5. The radius of the osculating circle is the same at every point t.

- ❑ true
- ❑ false

6. The osculating circle fills the coil.

- ❑ true
- ❑ false

7. Use a tech tool to graph and animate the normal vector, osculating circle, and curvature for r(t)=<cos t, sin t, t> as t moves through its domain of $0 \leq t \leq 10$.

ANSWERS: 2. r(t)=<cos t, sin t, t> 3. $0 \leq t \leq 10$ 4. T 5. T 6. T

ART: Arc Length Font 3 min

1. With your best handwriting, write your first name or any word in curvy, cursive letters.

cursive font

2. Now write your name in *Approximate Arc Length Cursive, i.e.,* as a computer calculating its approximate arc length with short linelets would.

linelet font 1

3. Do a few more approximations with linelets of longer length, then shorter length.

Clunky linelet font 3: longer linelets make a gross approximation.

Fancy linelet font 2: shorter linelets make a better approximation to cursive.

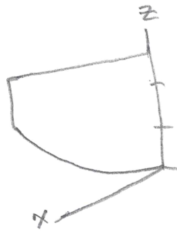
Each curved line consists of infinitely many straight lines, these themselves being infinitely small.
Marquis de L'Hopital

Curtain Match

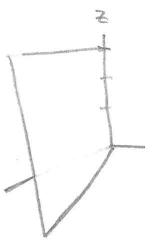
4 min

Match each image to the correct type of curtain integral.

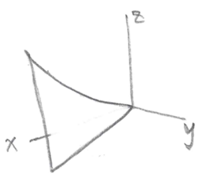
1.



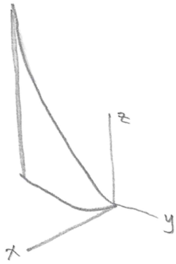
2.



3.



4.



A. curtain from plane curve to $z=f(x, y)$

$r(t)=\langle t, t, 0\rangle$ for $0 \leq t \leq 5$
$z=3$

B. curtain from plane curve to vertically stacked space curve

$r_1(t)=\langle t, t, 0\rangle$ for $0 \leq t \leq 5$
$r_2(t)=\langle t, t, .1t^2\rangle$ for $0 \leq t \leq 5$

C. curtain from space curve to vertically stacked space curve

$r_1(t)=\langle t, t, .1t^2\rangle$ for $0 \leq t \leq 5$
$r_2(t)=\langle t, t, .3t^2\rangle$ for $0 \leq t \leq 5$

D. curtain from space curve to $z=f(x, y)$

$r(t)=\langle t, t, .1t^2\rangle$ for $0 \leq t \leq 5$
$z=3$

ANSWERS: 1. D 2. A 3. B 4. C

Page Curtains

15 min

1. Tear a page from this journal. A good candidate is p. 5.

2. Manipulate (i.e., place or bend) this 5.5 by 8.5 inch page to represent each curtain integral below.

3. *Bonus:* Set up the curtain integral for each.

4. *Double Bonus:* Compute the curtain area for each integral. You already know the area, which is the area of the page (5.5 in × 8.5 in = 46.75 in^2). So this calculation allows you to verify that your modeling in the previous step is correct.

A. $\int_C 5.5\ |r'(t)|\ dt$, where the curve C is the upper semicircle $r(t)=\langle(8.5/\pi)\cos t, (8.5/\pi)\sin t\rangle$ for $0 \le t \le \pi$.

B. $\int_C 5.5\ |r'(t)|\ dt$, where the curve C is the parabola $r(t)=\langle t, t^2\rangle$ for $0 \le t \le 2.788$.

C. $\int_C 8.5\ |r'(t)|\ dt$, where the curve C is the parabola $r(t)=\langle t, t^2\rangle$ for $0 \le t \le 2.198$.

D. $\int_C 8.5\ |r'(t)|\ dt$, where the curve C is the line $r(t)=\langle t, 3t\rangle$ for $0 \le t \le 1.739$.

E. The area of the curtain from the surface $z=5.5$ to the exponential curve $r(t)=\langle t, e^t\rangle$ for $0 \le t \le 2.207$.

Appreciate that you can think of each curtain integral A through E in two ways. For example, curtain B goes from the parabolic plane curve in the xy-plane to either the surface $z=5.5$ **or** the space curve $r(t)=\langle t, t^2, 5.5\rangle$.

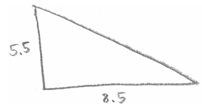
ANSWERS: next page

Flag Curtains

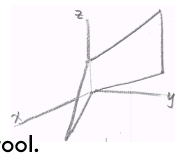
15 min

1. Tear a page from this journal. A good candidate is p. 5.

2. Make the page into a flag by cutting, or folding and tearing, along the diagonal.



3. Place this paper flag in your Desktop Coordinate System and manipulate it so that it represents each curtain described below. At this point in the journal, you should be able to visualize the surfaces $z=f(x, y)$ and curves $r(t)$ but it's okay if you need to use a tech tool.



4. Write the curtain integral that computes the area of each flag curtain. Verify that the integral you modeled is correct by using a tech tool to compute the area. You know, from geometry, that the area of each flag is $(5.5 \times 8.5)/2=23.3875$ in^2.

A. curtain from $z=8.5-(8.5/5.5)y$ to $r(t)=<0, 5.5-5.5t, 0>$ for $0\le t\le 1$.

B. curtain from $z=(5.5/8.5)x$ to $r(t)=<8.5-8.5t, 0, 0>$ for $0\le t\le 1$.

C. curtain from $r_1(t)=<8.5-8.5t, 0, 0>$ to $r_2(t)=<8.5-8.5t, 0, 5.5t>$ for $0\le t\le 1$.

D. piecewise smooth curtain:
piece 1 from $r_1(t)=<4.25-4.25t, 0, 0>$ to $r_2(t)=<4.25-4.25t, 0, 2.75t>$ for $0\le t\le 1$
piece 2 from $r_1(t)=<0, 4.25-4.25t, 0>$ to $r_2(t)=<0, 4.25-4.25t, 2.75t>$ for $1\le t\le 2$

E. curtain from $r_1(t)=<t, t^2, 0>$ to $r_2(t)=<t, t^2, (5.5/8.5)(.5t \sqrt{1+4t^2} + .25\sinh^{-1}(2t))>$ for $0\le t\le 2.788$.

F. curtain over the circle $r(t)=<(8.5/2\pi) \cos t, (8.5/2\pi) \sin t, 0>$ and under the helix $r(t)=<(8.5/2\pi) \cos t, (8.5/2\pi) \sin t, (5.5/2\pi)\, t >$ for $0\le t\le 2\pi$.

ANSWERS: next page

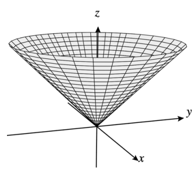
Surface Match

10 min

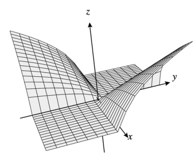
Use grid curves to (1) get an idea of the shape of each parametric function and (2) match each parametric function to an image.

1. $r(u, v) = \langle u, v, uv \rangle$
2. $r(u, v) = \langle \cos(v), \sin(v), u \rangle$
3. $r(u, v) = \langle 2\sin(v), 2\sin(v)\sin(u), 2\cos(v) \rangle$
4. $r(u, v) = \langle u, v, \sqrt{uv} \rangle$
5. $r(u, v) = \langle u\cos(v), u\sin(v), u \rangle$
6. $r(u, v) = \langle u, \sin(u)\cos(v), \sin(u)\sin(v) \rangle$

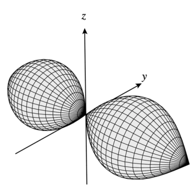
A.



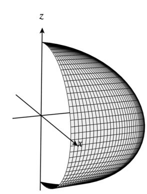
D.



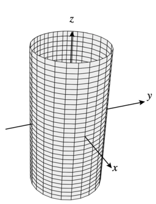
B.



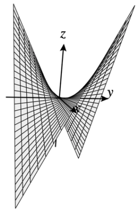
E.



C.



F.



ANSWERS: 1. F 2. C 3. E 4. D 5. A 6. B

Art: Create your own r(u, v)

10 min

Use a tech tool to create your own parametric surfaces. Experiment, trying to make something beautiful or something of interest to you.

Further Research: Mathematical surfaces have a long history in the architecture of roofs, from simple lean-to roofs and inverted V-roofs to domes and undulated roofs. The Spanish architect Antonio Gaudi, of the famous Sagrada Familia Cathedral in Barcelona, was known for his use of mathematical surfaces in his many designs. In fact, $r(u, v)=<u, v, au \sin(v/\mathbf{b})>$ is known as the Gaudi function, where the constants **a** and **b** affect the shape of the Gaudi surface. The "escoles" roof of the Sagrada Familia is a Gaudi function.

A few other architects that were also fond of mathematical surfaces are the Spanish-Mexican architect Felix Candela and the American-Canadian architect Frank Gehry.

Further Further Research: Make your own Calculus Paperweight or Calculus Desk Sculpture by learning how to send a parametrically defined surface to a 3D printer. The tech tool *CalcPlot3D* can create an STL file for 3D printing. See the Menu>Help>Creating STL files for 3D printing.

Hand Tangents

5 min

Suppose your hand from the wrist to the fingertips is a surface that can be represented by a parametric vector function.

Which vector function does the job?

- ❑ r(t)
- ❑ r(u, v)

Draw three dots on your hand as follows:

one point where the surface is particularly linear

one point where the surface is particularly nonlinear

any other point

Zoom in on each point several times.

The surface looks linear at each point so long as you zoom in enough.

- ❑ true
- ❑ false

For the highly nonlinear point, it takes more zooms for the tangent plane to look like the surface.

- ❑ true
- ❑ false

Place your pencil to represent the normal vector at each point.

Scavenger Hunt

15 min

Take a break. Walk outside. Give yourself a time limit, say 15 minutes. Mentally collect as many items from Chapter 2 as you see. For example, that laundry line in the neighbor's yard looks like a space curve. You can collect items and make a list when you return home or snap photos as you go that you annotate later.

1C. Think visually.