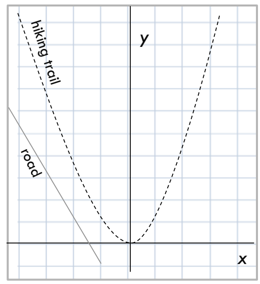
Challenge #1: hiking trail

A park ranger wants to add a path of shortest distance between the road and the hiking trail.



1. Which of the following best approximates the slope of the road?

   - ❑ 4
   - ❑ 2
   - ❑ −2
   - ❑ −4

2. Assume the hiking trail is well-approximated by the function $f(x)=x^2$. Draw some lines and make a guess as to which x-value on $f(x)=x^2$ is closest to the road. The slope of the tangent line at your guess is:

   - ❑ the same as the slope of the road.
   - ❑ perpendicular to the slope of the road.
   - ❑ steeper than the slope of the road.

3. In other (more mathematical) words, find the x-value of $f(x)=x^2$ at which the slope of the tangent line is:

   - ❑ 4
   - ❑ 2
   - ❑ −2
   - ❑ −4

4. So the park ranger should begin the path at the point (−1, 1) and continue until reaching the road, working at a slope of:

   - ❑ ¼
   - ❑ ½
   - ❑ −½
   - ❑ −¼

Do you want to try to "level up" on Challenge #1? Below are several upgrades.

Level 2 Suppose the road is approximated by the function $y=-2x-4$. The shortest path starts at the point $(-1, 1)$ on the trail. At what point on the road does it end?

Level 3 Suppose the road is approximated by the parametrized function $y=\mathbf{m}x-4$ with the parameter **m** between -3 and 0. Find, in terms of **m**, the point on the trail closest to the road. Verify your work by drawing a picture of your answer for three different values of **m**.

Level 4 Suppose the road is now given by the nonlinear function $y= -1 + \sqrt{-x-1}$. Find the point on the trail and the point on the road that gives the shortest path between them?

Challenge #2: vandal

Decartes' Most *Useful* Problem:

Finding a tangent line to a curve

"I dare say that this [finding a tangent to a curve] is not only the most useful and general problem that I know, but even that I have ever desired to know in geometry."

René Descartes (1596–1650) French

When Mr. Brown, owner of a hardware store, arrived at work Saturday morning, he immediately called the police. The famous outdoor statue of Venus that graced his storefront had been damaged in what appears to have been a drive-by paintball shooting. The police detective on the case reviewed the footage (both audio and video) from the surveillance camera of the convenience store on the opposite side of the road.

The function $f(t) = -\frac{1}{3}t^3 + 1.75\,t^2$ models the path of the road and the vandalized statue stands at the point (4, 7.5). From the footage the detective has determined that a loud thud occurred sometime during the t=3 and t=4 time frames. During that time frame, three vehicles (one at t=3, one at t=3.3 and one at t=3.4) passed by the curve in the road in front of the statue.

From which vehicle was the paintball gun fired?

Use calculus to determine which of the three vehicles was involved in the vandalism incident. Assume the paintball gun was aimed in the same direction that the vehicle was traveling. How confident are you in your answer? Could your solution method hold up in a court of law?

Hint #1: Graph the function f(t). Mark the locations of the statue and the three passing vehicles.
Hint #2: Draw the tangent line associated with each vehicle's location.

Do you want to try to "level up" on Challenge #2? Below are several upgrades.

Level 2 Suppose that the vandals actually had a more powerful paintball gun and shot from farther back on the road, somewhere between $t=0$ and $t=3$, before even reaching the curve by the shop. Assuming they still had the paint ball gun aimed in the vehicle's direction of travel, approximately where in this area was the vehicle when the shot was taken?

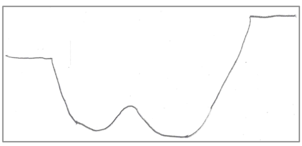
Level 3 Suppose the paintball guns have unlimited range and are still aimed in the vehicle's direction of travel. At how many points on the road, traveling East to West, could they hit the statue? What if they are allowed to travel in the other direction along the road?

Level 4 Extend the reasoning from Level 3. Is there a location the statue could be placed so that, even with infinite range, the vandals would never be able to hit it with their forward-facing, roof-mounted paint ball gun? Does the direction of travel along the road matter? Deduce a connection between the answer to that question and the range of values for the derivative.

Level 5 The paint ball gun is no longer mounted on the roof. Instead, one of the vandals is leaning out of the passenger window, giving him a 180 degree field of fire. That is, he can hit anything directly in front of the car, anything directly behind the car, and anything to the right. Also assume he has a maximum range of 8.5 units. When traveling East to West, find the stretch of road along which the vandal hits the statue. Do the same for traveling West to East. If the vandals want maximum time to pull off the perfect paintball shot, from which direction should they approach?

Challenge #3: skate park

A rule of the International Federation of Skate Parks states that *no park may have a ramp of steepness greater than 10 vertical units per 1 horizontal unit*. The local state park has a ramp with the profile below, which can be approximated by the piecewise function shown. A piecewise function is like a Frankenstein monster of many functions cut up and then stitched back together. Consider each piece of the function only over its corresponding interval of x-values.



On the left is a profile view of the skate ramp as seen from the side. The piecewise function that approximates the skate ramp is below.

$$y = \begin{cases} 10, & -8 < x \leq -2 \\ e^{-x+0.3}, & -2 < x \leq 3 \\ 2\sin(x+\pi/2)+2, & 3 < x \leq 9.5 \\ 0, & 9.5 < x \leq 15 \\ \tfrac{1}{4}(x-15)^3, & 15 < x \leq 19 \\ 16, & 19 < x \leq 24. \end{cases}$$

Does this park satisfy the Skate Federation rule? Use calculus to give a definitive answer.

❑ yes
❑ no

Try "leveling up" on Challenge #3. Below are several upgrades.

Level 2 If in Level 1 you answered that the skate park violates the regulation, then replace the offending section of the ramp with a different function that brings it in line. If in Level 1 you answered that the skate park passed the regulation, then extend one section of the ramp across more x-values so that the modified ramp now violates the regulation.

Level 3 Suppose a park is adamant about making a ramp that follows an exponential curve, $f(x)=a^x$, from $x=0$ to $x=2$. What is the maximum value of a that they could use and still satisfy the regulation?

Level 4 Suppose the skate park wants to design a new ramp that is a piecewise function built from a single function $g(x)$ and constant multiples of that function $c \times g(x)$. To insure the ramp is interesting, there must be at least four local maximums or minimums. However, it should also follow the regulation on steepness at every point. What are some good candidates for $g(x)$?

Level 5 Suppose we have an infinite skate ramp modeled with a single polynomial $f(x)$. Under what circumstances does $f(x)$ follow the regulation regarding steepness?

Challenge #4: murder mystery

Isaac Newton (1642–1727) English

Newton's Law of Cooling states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. Newton's Law of Cooling is represented by the formula

$$T = T_m + (T_0 - T_m)\, e^{-kt}$$

where

T = temperature of the object after t hours
T_m = temperature of the surrounding medium
T_0 = initial temperature of the object
k = rate of change in the temperature of the object
t = time, in hours, that the object has been exposed to the surrounding medium.

One application of Newton's law is to forensic medicine. A murder has occurred and the facts of the case are these. There are 5 suspects. Each has an airtight alibi, except for the following times each was alone during the day.

Miss Crimson: "home alone" from 10:30–11:45am
Professor Plumple: "stuck in traffic" from 11:45am–1:30pm
Mrs. Peabody: "baking cookies" from 1:30–3:45pm
Mr. Greely: "gone fishing" from 3:45–5:00pm
Col. Mustache: "can't recall" from 5:00–6:15pm

The butler has no alibi. So if the murder did not occur between 10:30am and 6:15pm, the butler must have done it!

The coroner arrived at the scene at 9:00pm and determined the temperature of the corpse to be 82.7°F. The room temperature is kept at exactly 65°F. One hour later, the body temperature has dropped to 80.8°F. Assuming the victim was healthy (before the murder, of course) and had a normal temperature of 98.6°F, who is the killer and when did the murder occur? Be precise to the second.

Why not "level up" on Challenge #4? Below are several upgrades.

Level 2 Be as precise with your Level 1 answer as you can. Give your answer in milliseconds. Then microseconds, then nanoseconds. Does this level of precision help your answer hold up in a court of law? Why or why not? Are there any other parts of your calculations that could use increased accuracy?

Level 3 Suppose Mr. Greely actually committed the murder at 3:45pm and tried to frame Colonel Mustache for it by raising the victim's body temperature with an electric blanket before leaving the scene at 5:00pm. By how much did Mr. Greely need to increase the victim's temperature before leaving?

Level 4 Back to the original Level 1 facts. Being a suspect and worried about her weak baking alibi, Mrs. Peabody decides to hire a lawyer, just to be safe. The lawyer's main argument revolves around the sensitivity of Newton's Law of Cooling. The lawyer claims that if the coroner's temperature reading is off by just 5%, then either Professor Plumple or Mr. Greely committed the murder. Do you agree? Can you prove your position?

Challenge #5: orbit

A satellite is moving clockwise in an orbit defined by $x^2+2y^2=4$. At any point on this orbit the satellite's thrusters can engage and the satellite can leave the orbit and move in the direction of the tangent at that point. At what point on the satellite's orbit must the thrusters be turned on in order to have the satellite connect with the docking station at point (3.5, .5)? This application requires extreme precision. How do you insure an extremely precise answer.

"Level up" on Challenge #5 with these upgrades.

Level 2 The satellite is moving along the same orbit but in the counterclockwise direction. At what point on the orbit should the thrusters engage to reach the docking station at (3.5, .5)?

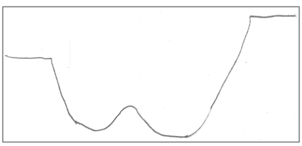
Level 3 There is some margin of error near the docking station. In fact, any satellite coming within a radius of .1 of the docking station at (3.5, .5) can dock. Give a range of points on the elliptical orbit traversed in the clockwise direction that now enable the satellite to make it to the docking station.

Level 4 Suppose that the satellite is used to capture images of 5 objects at locations ((0, 2), (2, .5), (3, −1), (−2, 2), and (1.5, −1)) outside the elliptical orbit. The camera of the satellite is pointed forward and can only capture objects in its direction of motion, i.e., along its tangent line. Assume the satellite is traveling in the clockwise direction. Give the 5 points on the elliptical orbit at which pictures should be taken.

Level 5 Begin with the Level 4 problem. Assume that the satellite is programmed to make one full orbit in the clockwise direction and another in the counterclockwise direction. Scientists want the clearest pictures possible and better pictures occur the closer the satellite is to the object. For each of the 5 objects, determine whether the picture should be taken from the clockwise or counterclockwise direction and identify the point at which to take the picture. Rank the 5 objects from best to worst in terms of clarity of their picture.

Challenge #6: skate park B

The International Federation of Skate Parks (from Challenge #2) has another rule. This one states that *no park may have a ramp of concavity greater than 10* in absolute value. The local skate park has a ramp with the profile below, which can be approximated by the piecewise function that follows. A piecewise function is like the Frankenstein monster: many functions cut up and then stitched back together. Consider each function only over its corresponding interval of x values.



On the left is a profile view of the skate ramp as seen from the side. The piecewise function that approximates the skate ramp is below.

$$y = \begin{cases} 10, & -8 < x \leq -2 \\ e^{-x+0.3}, & -2 < x \leq 3 \\ 2\sin(x+\pi/2)+2, & 3 < x \leq 9.5 \\ 0, & 9.5 < x \leq 15 \\ \tfrac{1}{4}(x-15)^3, & 15 < x \leq 19 \\ 16, & 19 < x \leq 24. \end{cases}$$

Does this park satisfy the Skate Federation rule? Use calculus to give a definitive answer.

❑ yes
❑ no

Do you want to try to "level up" on Challenge #6? Below are several upgrades.

Level 2 If in Level 1 you answered that the skate park violates the regulation, then replace the offending section of the ramp with a different function that brings it in line. If in Level 1 you answered that the skate park passed the regulation, then extend one section of the ramp across more x-values so that the modified ramp now violates the regulation.

Level 3 Suppose that the skate park wants to design a new ramp that is a piecewise function built from a single function $g(x)$ and constant multiples of that function $c \cdot g(x)$. Of course, the ramp should also follow the regulation on concavity at every point. What are some good candidates for $g(x)$?

Level 4 The skate park wants to use a logarithmic function $f(x) = \ln(g(x))$ for the section of the ramp between $x = 1/10$ and $x = 2$. What are some options for $g(x)$ that satisfy the regulation?

Level 5 Suppose we have an infinite skate ramp modeled after a single exponential function. Under what circumstances does it follow the regulation?

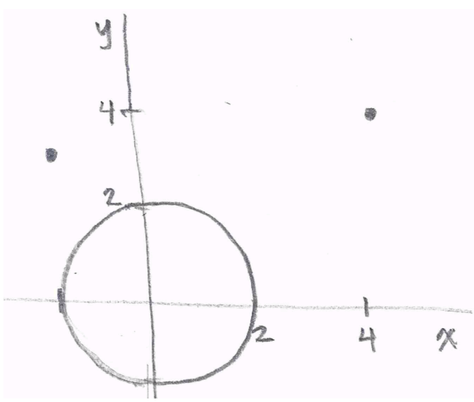
Challenge #7: suburbs

French mathematician Guillaume de L'Hopital (1661–1704)

L'Hopital's Problem:

Given two points outside a circle, find the point on the circle for which the sum of the two distances from the points to the circle is minimal.

Consider a suburban application of L'Hopital's problem. A city wants to place a new exit on the circular beltway around the city so that the sum of distances between the two suburbs (located outside the beltway) and the exit is minimal.



The NW suburb is located at (−2, 3), and the NE suburb is at (4, 4). The top half of the circular beltway is approximated by the equation $y=\sqrt{4-x^2}$. Place the new beltway exit so that the sum of distances between the suburbs and the exit is minimal. You can physically experiment to get an approximation. Make a few guesses on the figure below. Now confirm your guesses by using calculus to solve the problem exactly.

Further Research:
L'Hopital-Bernoulli controversy

Why not "level up" on Challenge #7? Below are several upgrades.

Level 2 Suppose the beltway is now elliptical and is approximated by the ellipse $\frac{1}{2}x^2+y^2=3$. Use the original two suburb points. Find the point on the circle for which the sum of the two distances from the two points to the **elliptical** beltway is minimal.

Level 3 In the original suburb problem the radius of the **circular** beltway is $r=2$. Solve a general version of this problem with a circle of radius r where $r<3$. Use the original suburb points of $(-2, 3)$ and $(4, 4)$. Find the point on the circle for which the sum of the two distances from the two points to the circle is minimal.

Level 4 Suppose the beltway is as originally described, a circle centered at the origin with a radius of 2. But now there is a third suburb point at $(2, 6)$. Find the point on the circle for which the sum of the three distances from the three points to the circle is minimal.

Level 5 Return to the Level 1 facts. Place the new exit on the circular beltway so that commuters from each suburb have the same drive length. In other words, the length of the commute from the NE suburb to the new exit must be the same as the length of the commute from the NW suburb to the new exit.

The most beautiful solid is the sphere and the most beautiful plane figure—the circle.
Pythagoras

Challenge #8: luge

Making a good luge (or bobsled or skeleton) course is not easy. The course must provide challenge at an appropriate level. A hard course poses safety issues; Georgian luger Nodar Kumaritashvili suffered a fatal crash in the 2010 Olympics in Whistler, British Columbia, Canada.

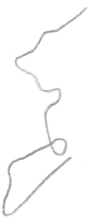
Use ideas from calculus to quantify the challenge level of the Whistler track using the 2D map shown to the left. Of course, this is a simplification since the actual track is 3D with elevation changes and gradients. (If you want to learn how to analyze the 3D track, continue onto Calculus 3.)

Zoomed view of the 2010 Whistler, Canada luge track

1984

Sarajevo, Bosnia Olympics Length ≈ 1200m

1988



Calgary, Canada Olympics Length ≈ 1250m

1994

Lillehammer, Norway Olympics Length ≈ 1350m

2008

Oberhof, Germany Worlds Length ≈ 1350m

Give these Challenge #8 upgrades a try.

Level 2 Even though you do not have a function to describe the track, you can still quantify directions of motion and how dramatically (or not) they change over time as the luger proceeds through the track. How? Make a chart or visualization.

Level 3 Even though you do not have a function to describe the track, you can still estimate the curvature with osculating circles. Incorporate this into your analysis from previous levels.

Level 4 Finally consider speed. While it is true that straight sections have no curvature and are safe in that sense, because the direction of motion is constant, the luger can quickly gain speed on these sections. How can you estimate and quantify speed and add this to your analysis from previous levels?

Level 5 Use your methods to order all 8 luge tracks from least to most challenging. Now create 8 more tracks by reversing the start and finish of the original 8 tracks. Order these 16 tracks from least to most challenging. Does it seem that tracks have gotten more challenging over the years?

2010

Whistler, Canada Olympics Length ≈ 1400m

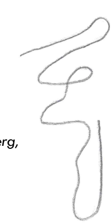
2012

Altenberg, Germany Worlds Length ≈ 1350m

2016

Koenigssee, Germany Worlds Length ≈ 1250m

2019



Winterberg, Germany Worlds Length ≈ 1300m